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Stability of Solutions of First-Order Impulsive Partial Differential Equations

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Theorems on stability and asymptotic stability of solutions of impulsive partial differential equations of first order are proved. These results are obtained via the method of differential inequalities and via the method of Lyapunov functions.

1. INTRODUCTION

Many evolution processes are characterized by the fact that at certain moments of time the parameters of the system are abruptly changed. An adequate mathematical apparatus for simulation of these processes and phenomena is the theory of impulsive differential equations. Numerous applications in theoretical physics, population dynamics, impulse techniques, biotechnology, robotics, etc., have led to its rapid development in recent years (Bainov *et al.*, 1989).

More recently the theory of impulsive partial differential equations has begun to emerge (Erbe *et al.*, 1991; Rogovchenko, 1988; Rogovchenko and Trofimchuk, 1986). This new theory gives greater possibilities for mathematical simulation of the above processes. The authors believe that it will undergo a rapid development in the coming years.

In this paper we consider the stability and asymptotic stability of solutions of impulsive partial differential equations of first order. We employ the method of differential inequalities and the method of Lyapunov functions.

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2. PRELIMINARY NOTES

Suppose that

$$\alpha = (\alpha_1, \ldots, \alpha_n)\colon \quad R_+ \to R^n,$$

$$\beta = (\beta_1, \ldots, \beta_n)\colon \quad R_+ \to R^n$$

are given functions and

$$\alpha(x) < \beta(x) \qquad \text{for} \quad x \in R_+$$

For two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ let the inequality $x \le y$ mean $x_i \le y_i$, $i = 1, \ldots, n$.

Let

$$E = \{(x, y) \in R^{1+n}\colon x \in R_+, \alpha(x) \le y \le \beta(x)\}$$

Assume that

$$0 = x_0 < x_1 < x_2 < \cdots < x_p < \cdots$$

are given numbers such that $\lim_{p\to\infty} x_p = \infty$.

We define

$$\Gamma_k = \{(x, y) \in E\colon x_k < x < x_{k+1}\}, \qquad k = 0, 1, \ldots; \quad \Gamma = \bigcup_{k=0}^{\infty} \Gamma_k$$

Let $C_{\text{imp}}[E, R]$ be the class of all functions $Z\colon E \to R$ such that:

(i) The functions $Z|_{\Gamma_k}$, $k = 0, 1, \ldots$, are continuous.

(ii) For each k, $k = 1, 2, \ldots$, $x = x_k$, there exists

$$\lim_{\substack{(t,s)\to(x,y)\\ t<x}} Z(t, s) = Z(x^-, y), \qquad \alpha(x) \le y \le \beta(x)$$

(iii) For each k, $k = 0, 1, \ldots$, $x = x_k$, there exists

$$\lim_{\substack{(t,s)\to(x,y)\\ t>x}} Z(t, s) = Z(x^+, y), \qquad \alpha(x) \le y \le \beta(x)$$

(iv) For each k, $k = 0, 1, \ldots$, $x = x_k$, we have

$$Z(x, y) = Z(x^+, y), \qquad \alpha(x) \le y \le \beta(x)$$

For a function $Z \in C_{\text{imp}}[E, R]$ we define

$$\Delta Z(x_k, y) = Z(x_k, y) - Z(x_k^-, y), \qquad y \in [\alpha(x_k), \beta(x_k)], \quad k = 1, 2, \ldots$$

Suppose that

$$f\colon \quad E \times R \times R^n \to R, \qquad \phi\colon \quad [\alpha(0), \beta(0)] \to R, \qquad g\colon \quad E \times R \to R$$

are given functions.

We consider the initial value problem (IVP)

$$Z_x(x, y) = f(x, y, Z(x, y), Z_y(x, y)) \tag{1}$$

$$Z(0, y) = \phi(y), \qquad y \in [\alpha(0), \beta(0)] \tag{2}$$

$$\Delta Z(x_k, y) = g(x_k, y, Z(x_k^-, y)), \qquad y \in [\alpha(x_k), \beta(x_k)] \tag{3}$$

where $k = 1, 2, \ldots$; $Z_y = (Z_{y_1}, \ldots, Z_{y_n})$.

Definition 1. A function $Z: E \to R$ is a solution of IVP (1)–(3) if:

(i) $Z \in C_{\text{imp}}[E, R]$, there exist derivatives $Z_x(x, y)$ and $Z_y(x, y)$ for $(x, y) \in \Gamma$, and Z satisfies (1) on Γ.
(ii) Z satisfies (2) and (3).
Let

$$S = \bigcup_{k=0}^{\infty} \{\partial\Gamma_k \cap [(x_k, x_{k+1}) \times R^n]\}$$

A function $Z \in C_{\text{imp}}[E, R]$ will be called a *function of class* $C^*_{\text{imp}}[E, R]$ if Z has partial derivatives $Z_x(x, y)$ and $Z_y(x, y)$ for $(x, y) \in \Gamma$ and there exists the total derivative of Z on S.

We define functions I_0, I_+, and I_- as follows. For each point $(x, y) \in E$ there exist sets of integers $I_0[x, y]$, $I_+[x, y]$, and $I_-[x, y]$ such that

$$I_0[x, y] \cup I_+[x, y] \cup I_-[x, y] = \{1, \ldots, n\}$$

and

$$y_i = \alpha_i(x) \qquad \text{for} \quad i \in I_-[x, y]$$

$$y_i = \beta_i(x) \qquad \text{for} \quad i \in I_+[x, y]$$

$$\alpha_i(x) < y_i < \beta_i(x) \qquad \text{for} \quad i \in I_0[x, y]$$

We define

$$P_k = (x_k, x_{k+1}), \qquad k = 0, 1, \ldots; \qquad P = \bigcup_{k=0}^{\infty} P_k$$

Let $C_{\text{imp}}[R_+, R]$ be the class of all functions $W: R_+ \to R$ such that:

(i) The functions $W|_{P_k}$, $k = 0, 1, \ldots$, are continuous.
(ii) For each k, $k = 1, 2, \ldots$, $x = x_k$, there exists

$$\lim_{\substack{t \to x \\ t < x}} W(t) = W(x^-)$$

(iii) For each k, $k = 0, 1, \ldots$, $x = x_k$, there exists

$$\lim_{\substack{t \to x \\ t > x}} W(t) = W(x^+)$$

(iv) For each k, $k = 0, 1, \ldots$, $x = x_k$, we have $W(x) = W(x^+)$.

We adopt the following definitions of stability:

Definition 2. The trivial solution $U \equiv 0$ of IVP (1)–(3) is said to be *stable* if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|\phi(y)| < \delta$, $y \in [\alpha(0), \beta(0)]$ implies

$$|U(x, y)| < \varepsilon \qquad \text{on} \quad E$$

Definition 3. The trivial solution $U \equiv 0$ of IVP (1)–(3) is said to be *asymptotically stable* if:

(i) It is stable.
(ii) There exists a positive number δ_0 such that for every $\varepsilon > 0$ there corresponds $X(\varepsilon)$ such that $|\phi(y)| < \delta_0$, $y \in [\alpha(0), \beta(0)]$, implies $|U(x, y)| < \varepsilon$ for $x \geq X(\varepsilon)$, $\alpha(x) \leq y \leq \beta(x)$.

We introduce the class of functions K.

Definition 4. A function a belongs to class K if $a \in C(R_+, R_+)$, $a(0) = 0$, and a is strictly increasing on R_+.

3. MAIN RESULTS

3.1. Estimates by Solutions of Ordinary Differential Equations

We introduce the following assumptions:

H1: $\sigma \in C(R_+ \times R_+, R_+)$.
H2. $\tilde{\sigma} \in C(R_+ \times R_+, R_+)$ is such that

$$\gamma(p) = p + \tilde{\sigma}(x, p)$$

is nondecreasing on R_+.

Lemma 1. Suppose that the following conditions hold:

1. Assumptions H1, H2 are met and $\varphi \in C_{\text{imp}}[R_+, R]$.
2. $\varphi(0) \leq \eta_0$, $\eta_0 \in R_+$.
3. $\omega(\cdot, \eta_0)$: $R_+ \to R_+$ is the maximum solution of the problem

$$W'(x) = \sigma(x, W(x)), \qquad x \in P \tag{4}$$

$$W(0) = \eta_0 \tag{5}$$

$$W(x_k) = W(x_k^-) + \tilde{\sigma}(x_k, W(x_k^-)), \qquad k = 1, 2, \ldots \tag{6}$$

4. For $x \in P$ we have

$$\mathscr{D}_- \varphi(x) \leq \sigma(x, \varphi(x)) \tag{7}$$

where $\mathscr{D}_-$ is the left-hand lower Dini derivative and

$$\varphi(x_k) \le \varphi(x_k^-) + \tilde{\sigma}(x_k, \varphi(x_k^-)), \qquad k = 1, 2, \ldots \tag{8}$$

Then we have

$$\varphi(x) \le \omega(x; \eta_0), \qquad x \in R_+ \tag{9}$$

We omit the proof of Lemma 1.

Introduce the following assumptions:

H3. $\alpha, \beta \in C(R_+, R^n)$ and $\alpha|_{p_k}$ and $\beta|_{p_k}$, $k = 0, 1, \ldots$, are of class C^1.
H4. For $x \in R_+$ we have $\alpha(x) < \beta(x)$.

Theorem 1. Suppose that the following conditions hold:

1. Assumptions H1–H4 are fulfilled.
2. $f \in C(E \times R \times R^n, R)$ and

$$\begin{aligned} f(x, y, p, q) \operatorname{sign} p - \sum_{i \in I_-[x,y]} \alpha_i'(x)|q_i| \\ + \sum_{i \in I_+[x,y]} \beta_i'(x)|q_i| \le \sigma(x, |p|) \end{aligned} \tag{10}$$

where $q_i = 0$ for $i \in I_0[x, y]$.

3. For $(x, y) \in E$ we have

$$f(x, y, 0, 0) = 0, \qquad g(x, y, 0) = 0$$

$$\sigma(x, 0) = 0, \qquad \tilde{\sigma}(x, 0) = 0$$

4. For $(x, y, p) \in E \times R$ we have

$$|g(x, y, p)| \le \tilde{\sigma}(x, |p|) \tag{11}$$

5. The maximum solution of (4)–(6) is defined on R_+.
6. The function $\phi \in C([\alpha(0), \beta(0)], R)$ and any solution U of IVP (1)–(3) is of class $C^*_{\text{imp}}[E, R]$.

Then, if the trivial solution of problem (4)–(6) is stable (asymptotically stable), then the trivial solution of IVP (1)–(3) is stable (asymptotically stable).

Proof. Suppose that the trivial solution of (4)–(6) is stable. Then for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $\eta_0 < \delta$, then

$$\omega_0(x; \eta_0) < \varepsilon, \qquad x \in R_+$$

where $\omega_0(\cdot\,; \eta_0)$ is a solution of (4)–(6).

We prove that if $|U(0, y)| \le \eta_0$ and $y \in [\alpha(0), \beta(0)]$, then

$$|U(x, y)| \le \omega(x; \eta_0), \qquad (x, y) \in E \tag{12}$$

$\omega(\cdot\,; \eta_0)$ is the maximum solution of (4)–(6).

We define the functions

$$\varphi(x) = \max_{y \in S_x} |U(x, y)|$$

$$M(x) = \max_{y \in S_x} U(x, y)$$

$$N(x) = \max_{y \in S_x} (-U(x, y))$$

where $S_x = \{y: (x, y) \in E\}$, $x \in R_+$.

Now we prove that φ satisfies

$$\mathscr{D}_- \varphi(x) \le \sigma(x, \varphi(x)), \qquad x \in P \tag{13}$$

Suppose that $\tilde{x} \in P$. It follows that there exists $\tilde{y} \in [\alpha(\tilde{x}), \beta(\tilde{x})]$ such that either

$$\varphi(\tilde{x}) = M(\tilde{x}) = U(\tilde{x}, \tilde{y}), \qquad \mathscr{D}_- \varphi(\tilde{x}) \le \mathscr{D}^- M(\tilde{x}) \tag{14}$$

or

$$\varphi(\tilde{x}) = N(\tilde{x}) = -U(\tilde{x}, \tilde{y}), \qquad \mathscr{D}_- \varphi(\tilde{x}) \le \mathscr{D}^- N(\tilde{x}) \tag{15}$$

where $\mathscr{D}^-$ is the left-hand upper Dini derivative.

Assume that (14) holds.

If $(\tilde{x}, \tilde{y}) \in \operatorname{Int} \Gamma$, then

$$U_y(\tilde{x}, \tilde{y}) = 0$$

and

$$\mathscr{D}^- M(\tilde{x}) \le U_x(\tilde{x}, \tilde{y}) \tag{16}$$

Then we get from (14) and (16) that

$$\mathscr{D}_- \varphi(\tilde{x}) \le U_x(\tilde{x}, \tilde{y}) = f(\tilde{x}, \tilde{y}, U(\tilde{x}, \tilde{y}), 0) \operatorname{sign} U(\tilde{x}, \tilde{y})$$
$$\le \sigma(\tilde{x}, \varphi(\tilde{x})).$$

If $(\tilde{x}, \tilde{y}) \in S$, then we have

$$\begin{aligned} U_{y_i}(\tilde{x}, \tilde{y}) &\le 0 \qquad \text{for} \quad i \in I_-[\tilde{x}, \tilde{y}] \\ U_{y_i}(\tilde{x}, \tilde{y}) &\ge 0 \qquad \text{for} \quad i \in I_+[\tilde{x}, \tilde{y}] \\ U_{y_i}(\tilde{x}, \tilde{y}) &= 0 \qquad \text{for} \quad i \in I_0[\tilde{x}, \tilde{y}] \end{aligned} \tag{17}$$

Let $\xi(x) = U(x, r(x))$, $x \in [0, \tilde{x}]$, where $r = (r_1, \dots, r_n)$ is defined by

$$r_i(x) = \alpha_i(x), \qquad i \in I_-[\tilde{x}, \tilde{y}]$$
$$r_i(x) = \beta_i(x), \qquad i \in I_+[\tilde{x}, \tilde{y}]$$
$$r_i(x) = \tilde{y}_i, \qquad i \in I_0[\tilde{x}, \tilde{y}]$$

Then we have

$$\begin{aligned} &\xi(x) \le \varphi(x), \qquad x \in [0, \tilde{x}] \\ &\xi(\tilde{x}) = \varphi(\tilde{x}) \end{aligned} \tag{18}$$

Therefore from (16)–(18) we have

$$\begin{aligned} \mathscr{D}_- \varphi(\tilde{x}) \le \mathscr{D}_- \xi(\tilde{x}) &= U_x(\tilde{x}, \tilde{y}) + \sum_{i \in I_-[\tilde{x},\tilde{y}]} \alpha_i'(\tilde{x}) U_{y_i}(\tilde{x}, \tilde{y}) \\ &\quad + \sum_{i \in I_+[\tilde{x},\tilde{y}]} \beta_i'(\tilde{x}) U_{y_i}(\tilde{x}, \tilde{y}) \\ &= f(\tilde{x}, \tilde{y}, U(\tilde{x}, \tilde{y}), U_y(\tilde{x}, \tilde{y})) \cdot \operatorname{sign} U(\tilde{x}, \tilde{y}) \\ &\quad - \sum_{i \in I_-[\tilde{x},\tilde{y}]} \alpha_i'(\tilde{x}) |U_{y_i}(\tilde{x}, \tilde{y})| \\ &\quad + \sum_{i \in I_+[\tilde{x},\tilde{y}]} \beta_i'(\tilde{x}) |U_{y_i}(\tilde{x}, \tilde{y})| \\ &\le \sigma(\tilde{x}, \varphi(\tilde{x})) \end{aligned}$$

In a similar way we prove (13) when (15) holds.
Let $x = x_k$ for some k, $k = 1, 2, \dots$. Then we have

$$\begin{aligned} \varphi(x_k) = |U(x_k, \tilde{y})| &\le |U(x_k^-, \tilde{y})| + |g(x_k, \tilde{y}, U(x_k^-, \tilde{y}))| \\ &\le \varphi(x_k^-) + \tilde{\sigma}(x_k, \varphi(x_k^-)), \qquad k = 1, 2, \dots \end{aligned} \tag{19}$$

Therefore all conditions of Lemma 1 are satisfied and we have that (12) holds, which proves the statement of the theorem. ■

3.2. Stability via Lyapunov Functions

Theorem 2. Suppose that the following conditions hold:

1. Assumptions H1–H4 are satisfied.
2. V is a function of the variables (x, p) defined on $R_+ \times R$.

3. V possesses continuous partial derivatives with respect to (x, p) such that for any solution $U \in C^*_{\text{imp}}[E, R]$ of IVP (1)–(3) we have

$$\frac{\partial V}{\partial x}(x, U(x, y)) + \frac{\partial V}{\partial p}(x, U(x, y)) \cdot f(x, y, U(x, y), U_y(x, y))$$

$$\leq G(x, y, V(x, U(x, y)), V_y(x, U(x, y))), \qquad x \neq x_k, \quad k = 1, 2, \ldots$$

$$\frac{\partial V}{\partial y}(x, U(x, y))$$

$$= \left(\frac{\partial V}{\partial p}(x, U(x, y)) \frac{\partial U}{\partial y_1}(x, y), \ldots, \frac{\partial V}{\partial p}(x, U(x, y)) \frac{\partial U}{\partial y_n}(x, y)\right)$$

$G \in C(E \times R \times R^n, R)$

4. For $(x, y, p, q) \in E \times R \times R^n$ we have

$$G(x, y, p, q) \operatorname{sign} p - \sum_{i \in I_-[x,y]} \alpha_i'(x)|q_i| + \sum_{i \in I_+[x,y]} \beta_i'(x)|q_i| \leq \sigma(x, |p|)$$

$$q_i = 0 \qquad \text{for} \quad i \in I_0[x, y]$$

5. For $y \in [\alpha(x_k), \beta(x_k)]$, $k = 1, 2, \ldots$, $p \in R$, we have

$$V(x_k, p + g(x_k, y, p)) - V(x_k, p) \leq \tilde{\sigma}(x_k, V(x_k, p))$$

6. The maximum solution $\omega(\cdot\,; \eta_0)$ of problem (4)–(6) is defined on R_+.

7. The function $\phi \in C([\alpha(0), \beta(0)], R)$ and $V(0, \phi(y)) \leq \eta_0$, $y \in [\alpha(0), \beta(0)]$.

Then any solution $U \in C^*_{\text{imp}}[E, R]$ of (1)–(3) satisfies

$$V(x, U(x, y)) \leq \omega(x; \eta_0) \qquad \text{on} \quad E \tag{20}$$

Proof. Let $m(x, y) = V(x, U(x, y))$. Then for $x \neq x_k$, $k = 1, 2, \ldots$, we have

$$\frac{\partial m}{\partial x}(x, y) = \frac{\partial V}{\partial x}(x, U(x, y)) + \frac{\partial V}{\partial u}(x, U(x, y)) \cdot f(x, y, U(x, y), U_y(x, y))$$

$$\leq G(x, y, m(x, y), m_y(x, y))$$

We have also

$$m(0, y) \leq \eta_0, \qquad y \in [\alpha(0), \beta(0)]$$

As in Theorem 1, we conclude that

$$m(x, y) \leq \omega(x; \eta_0), \qquad (x; y) \in E$$

that is,

$$V(x, U(x, y)) \leq \omega(x; \eta_0), \qquad (x, y) \in E. \quad \blacksquare$$

Theorem 3. Suppose that the following conditions hold:

1. The conditions of Theorem 2 are satisfied with

$$\begin{gathered} f(x, y, 0, 0) = 0, \qquad g(x, y, 0) = 0 \\ \sigma(x, 0) = 0, \qquad \tilde{\sigma}(x, 0) = 0; \qquad (x, y) \in E \end{gathered} \tag{21}$$

2. $b(|p|) \leq V(x, p) \leq a(|p|)$, where $a, b \in K$, $x \in R_+$, $p \in R$.

Then the stability or asymptotic stability of the trivial solution of (4)–(6) implies the stability or asymptotic stability of the trivial solution of IVP (1)–(3).

Proof. Suppose that the trivial solution of (4)–(6) is stable. Let $\varepsilon > 0$. Then given $b(\varepsilon) > 0$, there exists $\delta > 0$ such that $\eta_0 < \delta$ implies

$$\omega_0(x; \eta_0) < b(\varepsilon), \qquad x \in R_+ \tag{22}$$

where $\omega_0(\cdot\,; \eta_0)$ is a solution of (4)–(6).

By Theorem 2 we have

$$V(x, U(x, y)) \leq \omega(x; \eta_0), \qquad (x, y) \in E \tag{23}$$

for any solution $U \in C^*_{\text{imp}}[E, R]$ of (1)–(3), $\omega(\cdot\,; \eta_0)$ is the maximum solution of (4)–(6).

Choose a positive number $\delta_1 > 0$ such that $a(\delta_1) = \delta$ and assume that

$$|\phi(y)| \leq \delta_1, \qquad y \in [\alpha(0), \beta(0)]$$

This implies that

$$V(0, \phi(y)) \leq a(|\phi(y)|) \leq a(\delta_1) = \delta$$

Choose

$$\eta_0 = \sup_{y \in [\alpha(0), \beta(0)]} V(0, \phi(y))$$

From (21)–(23) it follows that

$$b(|U(x, y)|) \leq V(x, U(x, y)) \leq \omega(x; \eta_0) < b(\varepsilon)$$

which leads to $|U(x, y)| < \varepsilon$ on E, provided that $|\phi(y)| \leq \delta_1$, $y \in [\alpha(0), \beta(0)]$. This proves the stability of the trivial solution of IVP (1)–(3). It is easy to see that if the trivial solution of (4)–(6) is asymptotically stable, then the trivial solution of IVP (1)–(3) is asymptotically stable, too. $\blacksquare$

We introduce the following assumptions:

H5. The function $G: E \times R \times R^n \to R$ satisfies

$$G(x, y, p, q) - G(x, y, p, \bar{q}) + \sum_{i \in I_-[x,y]} \alpha_i'(x)(q_i - \bar{q}_i)$$

$$+ \sum_{i \in I_+[x,y]} \beta_i'(x)(q_i - \bar{q}_i) \leq 0$$

for $(x, y) \in S$, $p \in R$, $q = (q_1, \ldots, q_n)$, $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_n)$, and

$$q_i \leq \bar{q}_i \quad \text{for} \quad i \in I_-[x, y]$$

$$q_i \geq \bar{q}_i \quad \text{for} \quad i \in I_+[x, y]$$

$$q_i = \bar{q}_i \quad \text{for} \quad i \in I_0[x, y]$$

H6. The function $\delta(p) = p + g(x, y, p)$ is nondecreasing on R, $(x, y) \in E$.

H7. The function $\sigma: R_+ \times R_+ \to R_+$ is continuous, $\sigma(x, 0) = 0$ for $x \in R_+$ and the right-hand maximum solution of the problem

$$W'(x) = \sigma(x, W(x)), \qquad W(0) = 0$$

is $W(x) = 0$, $x \in R_+$.

H8. G satisfies the inequality

$$G(x, y, p, q) - G(x, y, \bar{p}, q) \geq -\sigma(x, \bar{p} - p)$$

for $p \leq \bar{p}$, $(x, y, p, q) \in E \times R \times R^n$.

H9. The function $\tilde{\sigma} \in C(R_+ \times R_+, R_+)$ is such that $\tilde{\sigma}(x, 0) = 0$, $x \in R_+$ and

$$g(x, y, p) - g(x, y, \bar{p}) \geq -\tilde{\sigma}(x, \bar{p} - p)$$

$p \leq \bar{p}$, $(x, y, p) \in E \times R$.

Theorem 4. Suppose that the following conditions hold:

1. Assumptions H3–H9 are fulfilled.
2. V is a function of the variables (x, p) defined on $R_+ \times R$.
3. V possesses continuous partial derivatives with respect to (x, p) such that for any solution $U \in C^*_{\text{imp}}[E, R]$ of IVP (1)–(3) we have

$$\frac{\partial V}{\partial x}(x, U(x, y)) + \frac{\partial V}{\partial p}(x, U(x, y)) \cdot f(x, y, U(x, y), U_y(x, y))$$

$$\leq G(x, y, V(x, U(x, y)), V_y(x, U(x, y))), \qquad x \neq x_k, \quad k = 1, 2, \ldots$$

$$\frac{\partial V}{\partial y}(x, U(x, y))$$
$$= \left(\frac{\partial V}{\partial p}(x, U(x, y)) \frac{\partial U}{\partial y_1}(x, y), \ldots, \frac{\partial V}{\partial p}(x, U(x, y)) \cdot \frac{\partial U}{\partial y_n}(x, y) \right)$$

4. For $y \in [\alpha(x_k), \beta(x_k)]$, $k = 1, 2, \ldots, p \in R$, we have

$$V(x_k, p + g(x_k, y, p)) - V(x_k, p) \le g(x_k, y, V(x_k, p))$$

Then, if $Z \in C^*_{\text{imp}}[E, R]$ is a solution of the problem

$$Z_x(x, y) = G(x, y, Z(x, y), Z_y(x, y)), \qquad (x, y) \in \Gamma \tag{24}$$

$$Z(0, y) = \psi(y), \qquad \psi \in C([\alpha(0), \beta(0)], R) \tag{25}$$

$$\Delta Z(x_k, y) = g(x_k, y, Z(x_k^-, y)), \qquad y \in [\alpha(x_k), \beta(x_k)], \qquad k = 1, 2, \ldots \tag{26}$$

and

$$V(0, \phi(y)) \le \psi(y), \qquad y \in [\alpha(0), \beta(0)]$$

we have

$$V(x, U(x, y)) \le Z(x, y) \qquad \text{on} \quad E$$

Proof. Let $m(x, y) = V(x, U(x, y))$, where U is any solution of (1)–(3), $U \in C^*_{\text{imp}}[E, R]$. Then we have

$$m_x(x, y) \le G(x, y, m(x, y), m_y(x, y)), \qquad (x, y) \in \Gamma$$
$$m(0, y) \le \psi(y), \qquad y \in [\alpha(0), \beta(0)]$$
$$\Delta m(x_k, y) \le g(x_k, y, m(x_k^-, y)), \qquad k = 1, 2, \ldots; \quad y \in [\alpha(x_k), \beta(x_k)]$$

Then all conditions of Theorem 2 from Bainov *et al.* (1994) are fulfilled and we conclude that

$$m(x, y) \le Z(x, y) \qquad \text{on} \quad E$$

that is,

$$V(x, U(x, y)) \le Z(x, y) \qquad \text{on} \quad E \quad \blacksquare$$

Theorem 4 may be used to obtain stability properties of the trivial solution of IVP (1)–(3).

Theorem 5. Let the conditions of Theorem 4 hold with $G(x, y, 0, 0) = 0$, $f(x, y, 0, 0) = 0$, and $g(x, y, 0) = 0$ for $(x, y) \in E$ and let $V(x, p)$ satisfy inequality (21).

Then the stability or the asymptotic stability of the trivial solution of (24)–(26) implies the stability or asymptotic stability of the trivial solution of IVP (1)–(3).

We omit the proof of Theorem 5.

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